

ECON 747 – LECTURE 4:
FROM COMPLETE TO INCOMPLETE MARKETS

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OVERVIEW OF THIS LECTURE

1. Why start with a complete markets model?
 2. Consumption and risk sharing with complete markets
 3. Asset pricing with complete markets
 4. The Lucas Tree model
 5. Asset pricing applications
- + Extra material on no-Ponzi vs. transversality conditions

PURPOSE OF THIS LECTURE

- ▶ Acquire tools that we need throughout the course (e.g. how to price assets)
- ▶ Build a benchmark model that models with financial frictions will deviate from
- ▶ Remember the two deviations from a complete markets representative agent model that are needed to meaningfully introduce financial frictions
 1. Incomplete markets
 2. Heterogeneous agents
- ▶ We will see today that one of them is not enough: (certain) heterogeneity between agents can be insured away if markets are complete

INSIGHTS FROM THE COMPLETE MARKETS MODEL

- ▶ If you understand the complete markets benchmark, you will realize some important aspects of many macro models
- ▶ Some examples ...

INSIGHTS FROM THE COMPLETE MARKETS MODEL

- ▶ A representative agent model actually relies on **lots of** trade in financial assets!
- ▶ Why?
- ▶ The representative agent model is isomorphic to a model in which a continuum of heterogeneous agents trade insurance (state-contingent claims) to eliminate all idiosyncratic risk!

INSIGHTS FROM THE COMPLETE MARKETS MODEL

- ▶ Models with financial frictions often feature several layers of market incompleteness
- ▶ Example of Kiyotaki-Moore style collateral constraint

$$b_t \leq \theta k_t$$

- ▶ Usually two types of market incompleteness!
 1. Bond is not state contingent
 2. Bond trade is limited by physical asset(great explanation in [Cao and Nie, 2017](#))

INSIGHTS FROM THE COMPLETE MARKETS MODEL

- ▶ In many models, some asset markets are complete, others are incomplete
- ▶ Example 1: friction in asset trade between two countries, complete markets (full insurance) within each country
- ▶ Example 2: friction in asset trade between households and firms, complete markets (full insurance) within household sector and within firm sector

INSIGHTS FROM THE COMPLETE MARKETS MODEL

- ▶ You may occasionally see in a seminar that someone has actually made some asset markets incomplete without noticing
- ▶ Someone in the audience will say something like: “you must be assuming that agents cannot insure away this problem”
- ▶ Lesson: always remind yourself about the market structure you are operating in, be conscious about everything you are (implicitly) assuming about asset trade

REFERENCES FOR LECTURES 4 AND 5

- ▶ This lecture and the next one are pretty close to some chapters in the Ljungqvist and Sargent text book *Recursive Macroeconomic Theory* (2nd edition)
 - ▶ Lecture 4: Chapters 8, 12, 13
 - ▶ Lecture 5: Chapters 16, 17
- ▶ The notation will be similar to theirs
- ▶ I will provide a few additional references throughout

CONSUMPTION AND RISK SHARING WITH COMPLETE MARKETS

ENVIRONMENT

- ▶ Households indexed by $i = 1, \dots, I$
- ▶ Preferences:

$$U_t^i = \mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{t-\tau} u(c_\tau^i)$$

- ▶ Technology: exogenous endowment y_t^i in period t
- ▶ Denote state of the world s_t
 - ▶ State is vector of realizations $\{y_t^1, \dots, y_t^I\}$
 - ▶ $\pi(s^t)$ is the probability of a history of states s^t
 - ▶ $\pi(s_{t+1}|s^t)$ is the probability of state s_{t+1} given history s^t

COMMENT ON NOTATION

- ▶ We make the state s_t explicit in the notation
- ▶ This is because we will work with state-contingent contracts
- ▶ But this model is not different from what we have seen so far: in the model of the previous two lectures, a state is a realization of (Z_t, V_t)

TWO WAYS TO FORMULATE COMPLETE MARKERS

1. Arrow-Debreu securities

- ▶ Contingent claims traded in period-0, exercised every period
- ▶ $q^0(s^t)$: time-0 price of asset that pays 1 unit of consumption if history s^t realizes
- ▶ $a^i(s^t)$: agent i 's holdings of this asset

2. Arrow securities

- ▶ One-period ahead contingent claims
- ▶ $Q(s_{t+1}|s^t)$: history- s^t price of asset that pays 1 unit of consumption if state s_{t+1} realizes next period
- ▶ $A^i(s_{t+1}|s^t)$: agent i 's holdings of this asset

contingent = every state can be contracted on

ARROW-DEBREU SECURITIES, MARKET OUTCOME

- ▶ The time-0 problem of the agent i is

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t))$$

subject to

$$c^i(s^t) = y^i(s^t) + a^i(s^t)$$
$$\sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t) a^i(s^t) = 0$$

- ▶ Interpretation of last constraint: buy and sell claims to a clearing house in period 0, remain with zero balance

OPTIMALITY

- ▶ Substitute out $a^i(s^t)$ and denote μ^i the Lagrange multiplier on the combined constraint
- ▶ The FOC w.r.t $c^i(s^t)$ is

$$\beta^t \pi(s^t) u'(c^i(s^t)) = q^0(s^t) \mu^i$$

- ▶ This condition holds for all i and all s^t

COMPETITIVE EQUILIBRIUM DEFINITION

- ▶ Define $c(s^t) = \sum_i c^i(s^t)$ and $y(s^t) = \sum_i y^i(s^t)$
- ▶ A **competitive equilibrium** in this economy is defined as a sequence of allocations $\{c(s^t)\}_{t=0}^{\infty}$ and prices $\{q^0(s^t)\}_{t=0}^{\infty}$ such that for all s^t :
 1. Markets clear: $c(s^t) = y(s^t)$
 2. Given the price system, the optimality conditions are satisfied

IMPLICATIONS OF COMPLETE MARKETS

1. Perfect risk sharing
2. Consumption smoothing

PERFECT RISK SHARING

- ▶ Combine the optimality condition for two agents i and j

$$\frac{u'(c^i(s^t))}{u'(c^j(s^t))} = \frac{\mu^i}{\mu^j} \quad \forall i, j, s^t$$

PERFECT RISK SHARING

- ▶ Solve for $c^i(s^t)$ and sum over i

$$c^i(s^t) = u'^{-1} \left\{ \frac{\mu^i}{\mu^j} u'(c^j(s^t)) \right\}$$

$$\sum_i c^i(s^t) = \sum_i u'^{-1} \left\{ \frac{\mu^i}{\mu^j} u'(c^j(s^t)) \right\}$$

- ▶ From market clearing, $\sum_i c^i(s^t) = y(s^t)$
- ▶ Therefore, we get an optimality condition for j 's consumption that only depends on the aggregate endowment!
- ▶ Idiosyncratic risk is insured away

CONSUMPTION SMOOTHING

- ▶ Combine the optimality condition for two states s^t and s^τ

$$\beta^{t-\tau} \frac{\pi(s^t) u'(c^i(s^t))}{\pi(s^\tau) u'(c^i(s^\tau))} = \frac{q^0(s^t)}{q^0(s^\tau)} \quad \forall i, s^t, s^\tau$$

- ▶ Desire to equate marginal utility of consumption across time and states, attenuated by probabilities and prices ($MRS = MRT$)

PLANNER SOLUTION

- ▶ Planner problem in this economy (no prices!)

$$\max \sum_i \omega^i \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t))$$

subject to

$$\sum_i c^i(s^t) = \sum_i y^i(s^t) \quad \forall s^t$$

- ▶ ω^i is the welfare weight on agent i

OPTIMALITY

- ▶ Planner FOC w.r.t. $c^i(s^t)$ is

$$\beta^t \omega^i \pi(s^t) u'(c^i(s^t)) = \lambda(s^t) \quad \forall s^t$$

- ▶ $\lambda(s^t)$, the Lagrange multiplier on the resource constraint, is a function of s^t

MARKET VS. PLANNER OUTCOME

- ▶ The above condition is the same as the market FOC if

$$\begin{aligned}\omega^i &= (\mu^i)^{-1} \\ \lambda(s^t) &= q^0(s^t)\end{aligned}$$

- ▶ The competitive equilibrium is a *particular* Pareto optimal allocation of resources
 - ▶ Welfare weight inversely related to shadow price on individual constraint
- ▶ Market prices reflect shadow prices on resources

ARROW SECURITIES, MARKET OUTCOME

- ▶ Agent i 's problem is

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c^i(s^t))$$

subject to

$$c^i(s^t) + \sum_{s^{t+1}} Q(s^{t+1}|s^t) A^i(s^{t+1}|s^t) = y^i(s^t) + A^i(s^t)$$

$$A^i(s^{t+1}) \geq -\bar{A}^i(s^{t+1})$$

- ▶ $A^i(s_{t+1}|s^t) > 0 \rightarrow$ saving, $A^i(s_{t+1}|s^t) < 0 \rightarrow$ borrowing
- ▶ $\bar{A}^i(s^{t+1})$ is agent i 's natural debt limit

THE NATURAL DEBT LIMIT

- ▶ In the formulation with Arrow securities, we need to impose some restriction on asset trade to prevent Ponzi schemes
- ▶ The *natural debt limit* is the weakest of such restrictions
- ▶ It is the value of the maximum amount that agent i can repay starting from the period, assuming consumption is zero forever
- ▶ We do not need such a restriction with Arrow-Debreu securities. It is only needed when budget constraints are formulated in a sequential way.

THE NATURAL DEBT LIMIT

- ▶ In the setting above, we impose

$$\bar{A}^i(s^{t+1}) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q^{t+1}(s^\tau) y^i(s^\tau)$$

- ▶ $q^{t+1}(s^\tau)$: Arrow-Debreu price in units of the state s^{t+1} consumption good
- ▶ RHS of above equation: maximum value that agent i can repay, assuming that her consumption is zero from $t + 1$ onwards
- ▶ In the setting above, each agent i faces one no-Ponzi constraints for each state that can occur next period
 - More detailed remarks on no-Ponzi conditions at the end of the lecture

OPTIMALITY

- ▶ The FOC w.r.t $c^i(s^t)$ is

$$\pi(s^t)u'(c^i(s^t)) = \gamma^i(s^t)$$

- ▶ The FOC w.r.t $A^i(s_{t+1}|s^t)$ is

$$\beta\gamma^i(s^t) = Q(s_{t+1}|s^t)\gamma^i(s^{t+1})$$

- ▶ These condition hold for all i and all s^t and s^{t+1}
- ▶ i has a Lagrange multiplier for each s^t

OPTIMALITY

- ▶ We can combine the two conditions above and show that the outcome is the same as with Arrow-Debreu securities (see slide 18) if

$$Q(s_{t+1}|s^t) = \frac{q^0(s^{t+1})}{q^0(s^t)}$$

- ▶ We can iterate on this relation to get

$$q^0(s^{t+1}) = q^0(s^0) \prod_{j=1}^t Q(s_j|s^{j-1})$$

SOME NOTES

- ▶ With Arrow securities, decisions can still be thought of as made in period-0
 - ▶ Re-optimizing would not change allocation because optimal plan is time consistent in this setting
- ▶ Defining a competitive equilibrium with Arrow securities requires specifying an initial asset distribution
 - ▶ For example: $A^i(0) = 0 \forall i$

RECURSIVENESS

- ▶ So far we did not require a recursive structure
- ▶ We can additionally assume that s_t follows a Markov process
- ▶ In that case we can formulate a Bellmann equation with a time-invariant value function
- ▶ Above we were able to characterize the full sequence of allocations in a setting that is not necessarily recursive

ASSET PRICING WITH COMPLETE MARKETS

PRICING ASSETS WITH COMPLETE MARKETS

- ▶ Take framework from above and normalize $q^0(s^0) = 1$
- ▶ Use Euler equation (consumption smoothing relation) to solve for the price

$$q^0(s^t) = \beta^t \frac{\pi(s^t) u'(c^i(s^t))}{u'(c^i(s^0))}$$

- ▶ This is the time-0 price of an asset that pays one unit of consumption in period t if history s^t occurs

PRICING SYNTHETIC ASSETS

- ▶ After we have priced $q^0(s^t)$, we can price any asset
- ▶ For example, a risk-free bond (non-state contingent) in period t after history s^t

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- ▶ For example, a risk-free bond (non-state contingent) in period t after history s^t

$$P_{B,t} = R_t^{-1} = \sum_{s_{t+1}|s^t} Q(s_{t+1}|s^t) = \sum_{s_{t+1}} \frac{q^0(s_{t+1})}{q^0(s^t)}$$

MORE ASSETS

- ▶ Risk-free consol in period 0

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$$P_0^{consol} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t)$$

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- ▶ Risk-free consol in period 0

$$P_0^{consol} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t)$$

- ▶ Stock with dividend stream $d(s^t)$

MORE ASSETS

- ▶ Risk-free consol in period 0

$$P_0^{consol} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t)$$

- ▶ Stock with dividend stream $d(s^t)$

$$P_0^{stock} = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t) d(s^t)$$

IMPORTANT NOTE

- ▶ The consumption plan remains the same whether or not we introduce these additional assets
- ▶ Trading in Arrow-Debreu / Arrow securities already allows agents to generate the respective payment streams

THE LUCAS TREE MODEL

THE LUCAS TREE MODEL

- ▶ Based on Lucas (1978) : *Asset pricing in an exchange economy*
- ▶ Purpose of this model is to price risky assets and to understand the relation to the marginal utility of consumption
- ▶ The setting features a representative agent
 - ▶ Idiosyncratic risk has been insured away via complete markets

ENVIRONMENT

- ▶ One consumer (or a large number of identical ones)
- ▶ Preferences: $U_t = \mathbb{E}_t \sum_{\tau=t}^{\infty} \beta^{t-\tau} u(c_{\tau})$
- ▶ Technology: endowment of one tree that pays fruit y_t (with Markov property)
- ▶ Price the following assets:
 - ▶ Shares in the tree, denoted a_t , at price p_t
 - ▶ Risk-free bonds b_t , pays gross return R_t at the beginning of the period

STATE VARIABLES

- ▶ What are the state variables in this economy?

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b_t, a_t, y_t

BELLMAN EQUATION

$$V(a_t, b_t, y_t) = \max_{c_t, a_{t+1}, b_{t+1}} \{u(c_t) + \beta \mathbb{E}_t V(a_{t+1}, b_{t+1}, y_{t+1})\}$$

subject to

$$c_t + b_{t+1} + p_t a_{t+1} = a_t y_t + R_t b_t + p_t a_t$$

BELLMAN EQUATION

$$V(a_t, b_t, y_t) = \max_{c_t, a_{t+1}, b_{t+1}} \{u(c_t) + \beta \mathbb{E}_t V(a_{t+1}, b_{t+1}, y_{t+1})\}$$

subject to

$$c_t + b_{t+1} + p_t a_{t+1} = a_t y_t + R_t b_t + p_t a_t$$

or

$$c_t + b_{t+1} + p_t (a_{t+1} - a_t) = a_t y_t + R_t b_t$$

OPTIMALITY

- ▶ Combine FOCs w.r.t c_t and b_{t+1} to get Euler equation

$$u'(c_t) = \beta R_t \mathbb{E}_t [u'(c_{t+1})]$$

OPTIMALITY

- ▶ Combine FOCs w.r.t c_t and a_{t+1} to get asset pricing equation for the tree

$$p_t = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + y_{t+1}) \right]$$

- ▶ This is a recursive equation in p_t
- ▶ This is basic the (consumption) CAPM equation

RISK PREMIUM

- ▶ Define the expected asset return as the price appreciation plus the payoff (dividend) relative to the current price

$$\mathbb{E}_t R_t^A = \mathbb{E}_t \left[\frac{p_{t+1} + y_{t+1}}{p_t} \right]$$

RISK PREMIUM

- ▶ Using the optimality conditions and the fact that $E(XY) = E(X)E(Y) + COV(X, Y)$ to obtain

$$\mathbb{E}_t R_t^A = R_t - \frac{cov(R_t^A, \frac{u'(c_{t+1})}{u'(c_t)})}{\mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]}$$

- ▶ The consumer likes assets that give a relatively high return when the relative marginal utility of consumption is high
 - ▶ Consumer is happy with a lower expected return as long as the return has the “right” correlation with her expected consumption patterns

EQUILIBRIUM

- ▶ A **competitive equilibrium** in this economy is defined as a sequence of allocations $\{c_t, a_t, b_t\}_{t=0}^{\infty}$ and prices $\{p_t, R_t\}_{t=0}^{\infty}$ such that for all s^t :

1. Markets clear:

$$c_t = y_t$$

$$b_t = 0$$

$$a_t = 1$$

2. Given the price system, the optimality conditions are satisfied

GE ASSET PRICING

- ▶ When we impose equilibrium we can obtain asset prices (or relationships between asset prices) as a function of the model primitives
- ▶ The general equilibrium asset pricing relationship in the setting above is then

$$\mathbb{E}_t R_t^A = R_t - \frac{\text{cov}(R_t^A, \frac{u'(y_{t+1})}{u(y_t)})}{\mathbb{E}_t \left[\frac{u'(y_{t+1})}{u'(y_t)} \right]}$$

THE ILLUSION OF CHOICE

- ▶ Price mechanism provides agent with the illusion of choice: prices adjust so that agent consumes own fruit and does not save/borrow
- ▶ In this economy, quantities are trivial but prices are not
- ▶ Share and bond trade “do not matter” for consumption plan

ASSET PRICING APPLICATIONS

HOW TO PRICE ASSETS?

1. Define environment: preferences, technology, market structure
2. Solve agents' maximization problems (illusion of choice)
3. Only then apply market clearing to get general equilibrium asset pricing relationships

APPLICATION 1: THE TERM STRUCTURE

- ▶ From Euler equation

$$R_{1,t}^{-1} = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]$$

- ▶ Two period bond

APPLICATION 1: THE TERM STRUCTURE

- ▶ From Euler equation

$$R_{1,t}^{-1} = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]$$

- ▶ Two period bond

$$R_{2,t}^{-1} = \beta^2 \mathbb{E}_t \left[\frac{u'(c_{t+2})}{u'(c_t)} \right]$$

$$R_{2,t}^{-1} = \beta^2 \mathbb{E}_t \left[\frac{u'(c_{t+2})}{u'(c_t)} \frac{u'(c_{t+1})}{u'(c_{t+1})} \right]$$

$$R_{2,t}^{-1} = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \beta \frac{u'(c_{t+2})}{u'(c_{t+1})} \right]$$

$$R_{2,t}^{-1} = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} R_{1,t+1}^{-1} \right]$$

APPLICATION 1: THE TERM STRUCTURE

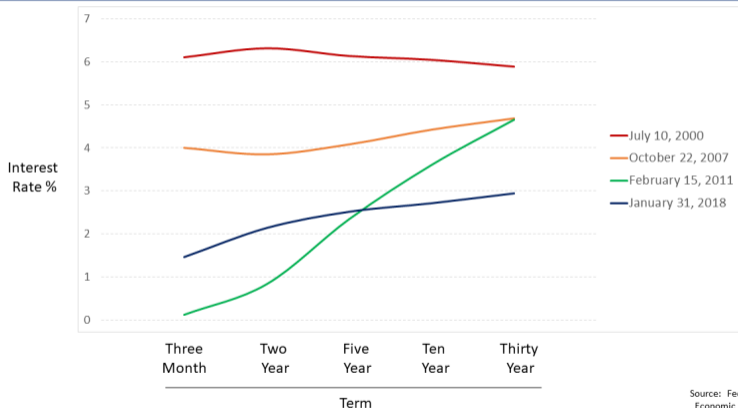
- ▶ If we assume risk neutrality $u(c) = c$, it follows that

$$R_{2,t} = R_{1,t} \mathbb{E}_t R_{1,t+1}$$

- ▶ The return on a two-period bond is the expected cumulative return on two one period bonds
- ▶ “Expectations hypothesis” result in finance
- ▶ Does not hold in practice, but useful to think about empirically observed yield curve in deviations from this benchmark (term premium, risk premium)

APPLICATION 1: THE TERM STRUCTURE

U.S. Treasury Yield Curves



Source: Federal Reserve
Economic Data (FRED)

APPLICATION 2: THE EQUITY PREMIUM PUZZLE

- ▶ Use Lucas' model asset pricing relationship with CRRA preferences

$$\mathbb{E}_t R_t^A = R_t - \frac{\text{cov} \left(R_t^A, \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \right)}{\mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \right]}$$

- ▶ If we think of R_t^A as the return on US equity, we can compute from the data whether this relation makes sense in practice

APPLICATION 2: THE EQUITY PREMIUM PUZZLE

- ▶ In the data

$$\begin{aligned}R_t^A &\approx 1.07 \\R_t &\approx 1.01 \\ \frac{c_{t+1}}{c_t} &\approx 1.02\end{aligned}$$

- ▶ Pick ex-ante admissible values for σ and use the data to compute LHS and RHS of the above relation. Does the equation hold?
- ▶ Not at all: this is the insight of [Mehra and Prescott \(1985\)](#)

APPLICATION 2: THE EQUITY PREMIUM PUZZLE

R. Mehra and E.C. Prescott, The equity premium

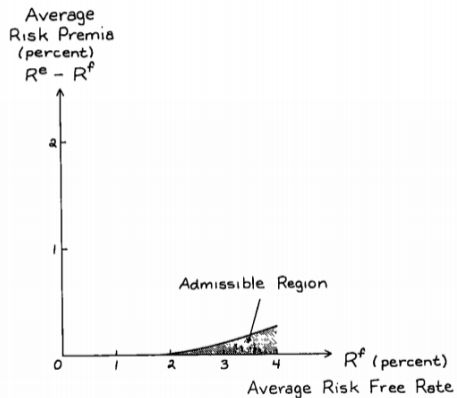


Fig. 4. Set of admissible average equity risk premia and real returns.

APPLICATION 2: THE EQUITY PREMIUM PUZZLE

- ▶ Proposed solutions to the equity premium puzzle
 - ▶ Separate IES from risk aversion (e.g. Epstein-Zin preferences)
 - ▶ Habits in consumption → [Campbell and Cochrane \(1999\)](#)
 - ▶ Rare disasters → [Barro \(2006\)](#)
 - ▶ Behavioural finance (Thaler and others)
 - ▶ Long-run risk → [Bansal and Yaron \(2004\)](#)

- ▶ See also [Mehra \(2007\)](#) for a summary paper

APPLICATION 3: PRICING PRODUCTIVE CAPITAL

- ▶ The risky assets we have priced above were endowment technologies
- ▶ We can also do asset pricing in the presence of production technology
- ▶ In particular, we can price productive capital that is installed in a firm
- ▶ This is the “Q” theory of investment

APPLICATION 3: PRICING PRODUCTIVE CAPITAL

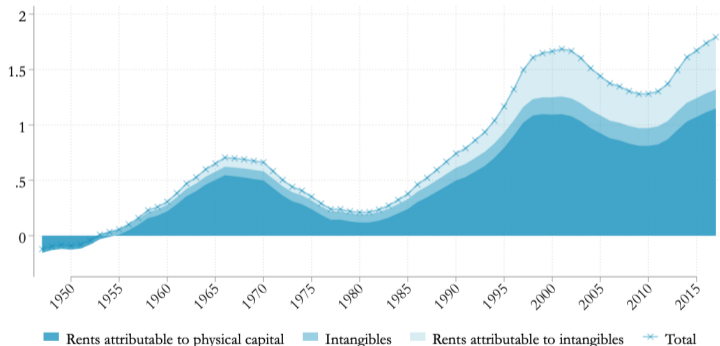
- ▶ Q is a relative price, the market value of capital (the value “inside the firm”) relative to the replacement value
- ▶ In a neoclassical model, Q is the shadow price on the investment accumulation equation (see e.g. [Hayashi, 1982](#))
- ▶ Important: *marginal* Q and *average* Q are different
- ▶ **Deriving and interpreting Q formally will be part of your second assignment**

APPLICATION 3: PRICING PRODUCTIVE CAPITAL

- ▶ In theory, a firms' marginal Q should be the sole predictor of investment
- ▶ In the data, marginal Q is hard to measure
- ▶ In the data, other determinants matter a lot for firm investment
 - ▶ For example financial constraints!
 - ▶ Plenty of empirical research, going back to [Fazzari, Hubbard, and Petersen \(1988\)](#)
- ▶ See also the literature on *dynamic corporate finance*, as surveyed by [Strebulaev and Whited \(2011\)](#)
 - ▶ Neoclassical models of various firm decision margins

A RECENT APPLICATION

- ▶ **Crouzet and Eberly (2020)** construct a decomposition of the gap between valuation (\approx average Q) and investment (\approx marginal Q) in the US



TAKING STOCK

TAKING STOCK

- ▶ We have build a benchmark model with complete markets
- ▶ We have priced assets
- ▶ The next lecture will start putting our necessary ingredients together:
Incomplete markets, heterogeneous agents and precautionary savings

EXTRA MATERIAL ON NO-PONZI VS. TRANSVERSALITY
CONDITIONS

NO-PONZI CONDITIONS

- ▶ Ruling out Ponzi schemes is
 1. A commonsense requirement economically
 2. In many settings a formal restriction that ensures the existence of a solution, as it *bounds utility*
- ▶ Above, we have formalized the no-Ponzi condition as a state-by-state inequality, using Arrow-Debreu prices
- ▶ Typically, no-Ponzi conditions are expressed as limits
- ▶ No-Ponzi conditions are used to consolidate budget constraints
- ▶ No-Ponzi conditions are conceptually different from transversality conditions

NO-PONZI CONDITIONS

- ▶ Suppose the following:
 - ▶ Agent starts with b_0
 - ▶ Has sequential budget constraint $c_t + b_{t+1} = Rb_t$
 - ▶ A candidate solution to her problem is $\{c_t^*\}_{t=0}^{\infty}$
- ▶ Without additional constraints, the agent could
 1. Choose $\tilde{c}_0 = c_0^* + 1$ and $\tilde{b}_1 = b_1^* - 1$
 2. For $t \geq 1$ choose $\tilde{c}_t = c_t^*$ and $\tilde{b}_{t+1} = b_{t+1}^* - R^t$
- ▶ This strategy satisfies the period-by-period constraint
- ▶ It is possible for *any* c_0^* , so there is no finite solution!

NO-PONZI CONDITIONS

- ▶ This situation is ruled out by adding the condition

$$\lim_{t \rightarrow \infty} \frac{b_t}{R^t} \geq 0$$

- ▶ This means that “terminal” asset holdings cannot be negative
- ▶ In the presence of this condition the agent cannot choose a solution that implies unbounded consumption/utility
- ▶ There can of course be stronger restrictions that make the no-Ponzi condition redundant, for example

$$b_t \geq 0 \quad \forall t$$

CONSOLIDATING THE BUDGET CONSTRAINT

- ▶ Start with constraint in period 0 and iterate to period T

$$\sum_{t=0}^T \left(c_t \frac{1}{R^t} \right) + \frac{b_{T+1}}{R^T} = Rb_0$$

- ▶ Doing this until infinity gives

$$\lim_{T \rightarrow \infty} \left\{ \sum_{t=0}^T \left(c_t \frac{1}{R^t} \right) + \frac{b_{T+1}}{R^T} \right\} = Rb_0$$
$$\sum_{t=0}^{\infty} \left(c_t \frac{1}{R^t} \right) + \lim_{T \rightarrow \infty} \frac{b_{T+1}}{R^T} = Rb_0$$

- ▶ Together with no-Ponzi condition, this gives

$$\sum_{t=0}^{\infty} \left(c_t \frac{1}{R^t} \right) \leq Rb_0$$

TRANSVERSALITY CONDITIONS

- ▶ Transversality conditions ensure the *sufficiency* of a solution
- ▶ We do not impose the transversality condition on the agent, but the agent will require this condition as part of her solution
- ▶ It is a prescription how to behave optimally, given a choice set
- ▶ It is needed because the solution to second-order difference require an initial and a terminal condition
 - ▶ Recall the remarks in Lecture 2

TRANSVERSALITY CONDITIONS

- ▶ Think of an Euler equation with substituted budget constraint:

$$u'(Rb_t - b_{t+1}) = \beta Ru'(Rb_{t+1} - b_{t+2})$$

- ▶ This is a second order difference equation in b_t
- ▶ The full solution to the agent's problem requires an initial condition (b_0 given) and the transversality condition

$$\lim_{t \rightarrow \infty} \frac{b_t}{R^t} \leq 0$$

- ▶ This says that the agent does not want to have savings in the limit
- ▶ It turns out this the same equation as the no-Ponzi condition, but with the weak inequality going the other way

NO-PONZI VS. TRANSVERSALITY

▶ **No-Ponzi condition**

- ▶ Ensures existence (bounds utility)
- ▶ Imposed on agent's program
- ▶ Only needed in competitive solution

▶ **Transversality condition**

- ▶ Ensures sufficiency (optimality)
- ▶ Comes out as part of agent's program's solution
- ▶ Part of competitive and planner solution

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